DIFFERENTIAL TECHNIQUES AND INFINITE REGIONS

B. H. McDonald
A. Wexler

University of Manitoba, Winnipeg, Canada

Abstract

Local solutions of unbounded static and harmonic problems are obtained using a variational formulation of the partial differential equations, with the trial functions constrained by the integral equations for the exterior field.

Introduction

Differential methods (eg. finite difference or finite element) for the solution of partial differential equations (P.D.E.), are very useful in small, closed regions. For unbounded regions, an integral equation (I.E.) formulation, which may be viewed as an "action-at-a-distance" method, is often most useful. In problems where the spatial dimensions are large, but where local "picture-frame" snapshots of the field are required it is useful to combine the P.D.E. techniques with the I.E. formulations. One may treat each picture frame as a closed, small region and solve the field therein using a P.D.E. method, with the I.E. used: (a) to represent the field outside the picture-frames; and (b) to link the picture-frames one to another. The I.E. may also be used to link the picture-frames to far field conditions, such as 1/r dependence at large distances.

Theoretical Considerations and Previous Work

To see how a picture-frame P.D.E. solution can be coupled to an I.E. formulation one may consider the simple configuration in Figure 1.

The problem in words is: given a potential on the boundary S_o of a conductor find the field in all space external to S_o. If this boundary potential is static, the P.D.E. to be solved is Laplace's, if the potential is harmonic, a Helmholtz operator must be considered.

The contour S_2 is chosen as the boundary of the picture-frame. The field between S_o and S_2 is to be obtained using a P.D.E. method, but additional information is required, namely the boundary condition on S_2.

Another contour, labelled S_1, is chosen as shown in Figure 1. The field exterior to S_1 is obtained using an I.E. formulation. For Laplace's equation, in two dimensions, one may write:

$$\phi(p) = \oint_{S_1} G(p|p') \left.\frac{\partial\phi}{\partial n}\right|_{S_1} ds' \qquad (1)$$

where:

$$G(p|p') = -\frac{1}{2\pi} \ln |p' - p| \qquad (2)$$

The virtual source distribution on S_1, from equation (1) is $\partial\phi/\partial n|_{S_1}$, and is, in fact, a function of the solution of the P.D.E. method indicated above. In particular, point p may be taken to be on S_2, giving a linear relationship between the field values on S_1 and S_2, which relates the picture-frame solution to the unbounded region. Similar arguments hold for the Helmholtz operator.

Clearly, this discussion is not restricted to only one picture-frame. As many snapshots as required may be obtained, provided that the contour S_1 is suitably chosen, with 'cuts' between picture-frames, so that S_1 becomes a single closed contour in each picture-frame, with the direction of integration fixed by the 'cuts'.

When the picture-frame solutions are available, the field is known in all space, since equation (1) may be applied then for any point p in all space exterior to S_1.

Several related schemes have recently been reported, which, in the main: (a) have used finite differences for the picture-frame solution; (b) have been restricted to one picture-frame; (c) have been restricted to Laplace's equation; and (d) have had to cope with singularities of Green's functions [1].

Silvester and Cermak [2]-[3], in effect, choose S_1 to coincide with S_2 (within one grid point in the finite difference representation). A potential shift operator is used to eliminate the $\partial\phi/\partial n$ term in equation (1). The resulting finite difference problem is solved by successive over-relaxation (S.O.R.). Since S_1 and S_2 are virtually coincident, the singularity of the Green's function occurring in equation (1) must be taken care of.

More recently Silvester and Hsieh [4] have applied finite elements, using a functional which effectively minimizes the energy in all space, including that exterior to S_2. This approach seems to be restricted to the Laplace operator (where total energy is finite), and the problem of singularity in the Green's function remains.

Sandy and Sage [5], in effect, choose S_1 to lie along the boundary of the conductor, S_o, and use an iterative procedure to select the correct potentials for S_2. Problems with singularities in the Green's function do not occur since potentials are not evaluated at source points. Problems can arise with convergence, and also, the computing time can be high due to the iterative procedures involved (including S.O.R.). Also, if any media inhomogeneities appear between S_1 and S_2, equation (1) becomes somewhat more involved.

Greenspan and Werner [6] discuss a finite difference approach for the Helmholtz operator, and show that solutions do exist, are unique, and can be numerically obtained. The contour S_1 is taken coincident with S_2, yielding singularity problems. Special functions are found to represent a Dirichlet boundary condition on S_2 and an error analysis is given.

The Present Scheme

The present scheme was designed: (a) to avoid singularities in the Green's functions; (b) to use the simplest form of the integral equations; (c) to handle inhomogeneous and anisotropic media (possible with existing finite element software); and (d) to avoid iterative procedures.

The picture-frame boundary S_2 and the contour S_1 are both chosen to include all inhomogeneities and anisotropies. The contour S_1 is chosen to lie well within the picture-frame boundary S_2, thus sidestepping the singularity problem of the Green's function. Since field values on both S_1 and S_2 are described by the trial functions of the finite element scheme, equation (1) provides the linear constraint which completes the P.D.E. solution.

Examples

Results are given for three simple experiments, demonstrating the scheme, and giving some indication of its potential.

For these examples, the z-directed electric vector potential is used [7], yielding equipotential lines of H_z, the contours being in the same direction as the electric field.

Figure 3 shows a plot of equipotential contours for a static problem with two conducting plates (infinite in the z-direction) with a line dipole source connected across the plates at $x = 0$.

Figure 4 shows the real part of the solution obtained (for the configuration of Figure 3), for the source at zero phase at 19 GHz.

Figure 5 shows the real part of the solution for an antenna type problem, with dielectric obstacles. Here the plot is for the source at zero phase at 5 GHz. Here two separate picture-frames are used, with a less accurate second order polynomial approximation.

The illustrations are the same as plots obtained by solving the basic finite element problem with "correct" (Dirichlet) potentials specified at the picture-frame boundaries. In these examples, however, no Dirichlet potentials were specified, the linear conditions were obtained from the I.E. formulation, equation (1).

The picture-frames are clearly snapshots of the entire field. Theoretically, the borders of the picture-frames are transparent to the final solutions. In practice, however, errors do occur. But the error at the picture-frame boundary is no worse than the error at any triangle interface.

More than one picture-frame may be used, and the solution is continuous in each picture-frame, and between them.

The scheme is clearly applicable to complicated microstrip problems, e.g., the coupling of adjacent microstrip lines. A microstrip example will be presented.

Computer times are within thirty percent of the above-mentioned basic finite-element solution of the equivalent, specified, Dirichlet problem. Accuracy depends upon the numerical methods for integrating equation (1), and also upon the parameters affecting finite-element solutions, namely the size, and number of elements, and the order of the polynomial trial functions.

Conclusions

The present scheme permits placement of picture-frames in regions where detailed analysis is required, and saves work and computer storage for the remaining "empty space". Nevertheless the solution anywhere can be constructed from the picture-frame solutions using a Green's function integration. The software at hand permits analysis of inhomogeneous and anisotropic regions, when these can all be placed in picture-frames. Of great algorithmic convenience is the fact that the problem of singularity of the Green's function is side-stepped by not evaluating potentials at source points.

References

[1] F.S. Acton, Numerical Methods that Work, Harper & Row, 1970, pp. 410 - 430.

[2] I.A. Cermak and P. Silvester, "Solution of two-dimensional field problems by boundary relaxation", Proc. Inst. Elect. Eng. (London), Vol. 115, Sept. 1968, pp. 1341 - 1348.

[3] P. Silvester and I.A. Cermak, "Analysis of coaxial line discontinuities by boundary relaxation", IEEE Trans. Microwave Theory Tech., Vol. MTT-17, Aug. 1969, pp. 489 - 495.

[4] P. Silvester and M.S. Hsieh, "Finite-element solution of 2-dimensional exterior-field problems" Proceedings IEE, Vol. 118, No. 12, Dec. 1971, pp. 1743 - 1747.

[5] F. Sandy and J. Sage, "Use of finite difference approximations to partial differential equations to problems having boundaries at infinity", IEEE Trans. Microwave Theory Tech. (Lett.), Vol. MTT-19, May 1971, pp. 484 - 486.

[6] D. Greenspan and P. Werner, "A numerical method for the exterior Dirichlet problem for the reduced wave equation", Archive for Rational Mechanics and Analysis, Vol. 23, 1966, pp. 288 - 316.

[7] R.F. Harrington, Time Harmonic Electromagnetic Fields, McGraw Hill (1961), pp. 129 - 131.

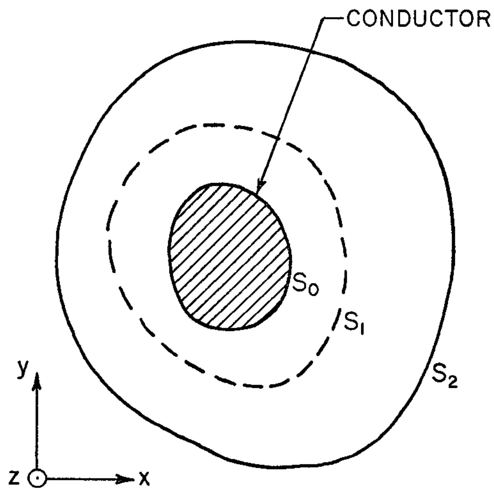

FIG. 1 THE PICTURE-FRAME AND ASSOCIATED CONTOURS

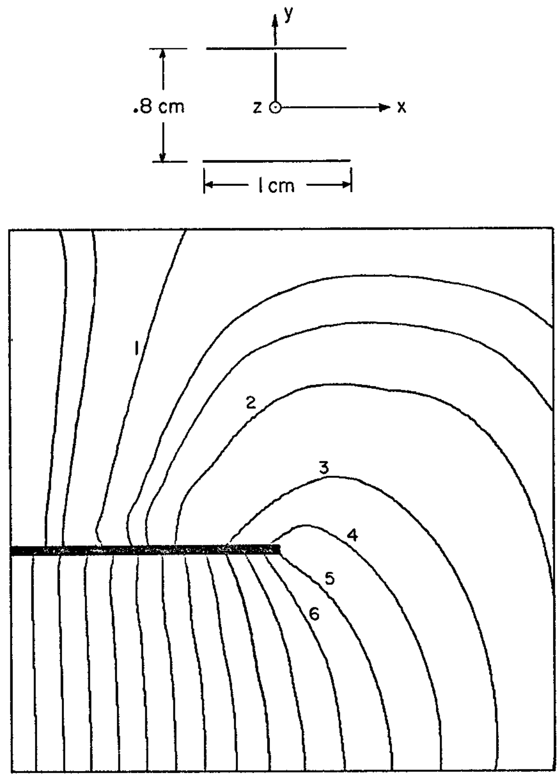

FIG. 2 A STATIC CAPACITOR PROBLEM AND SOLUTION USING ONE PICTURE-FRAME

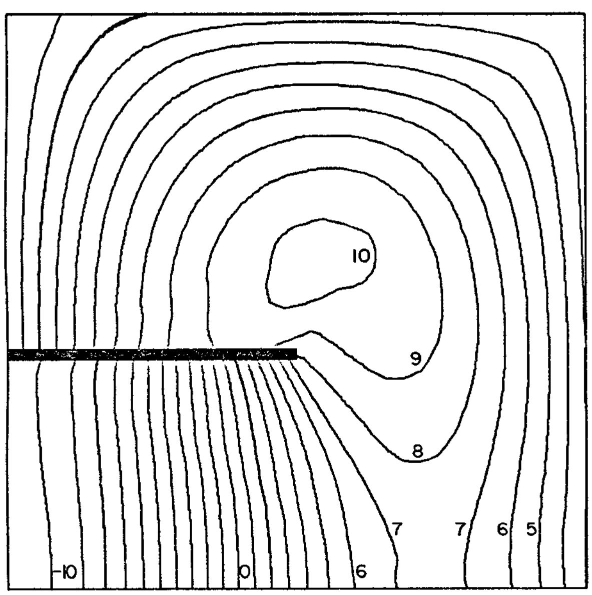

FIG. 3 A HARMONIC CAPACITOR SOLUTION, GEOMETRY AS IN FIGURE 3, USING ONE PICTURE-FRAME

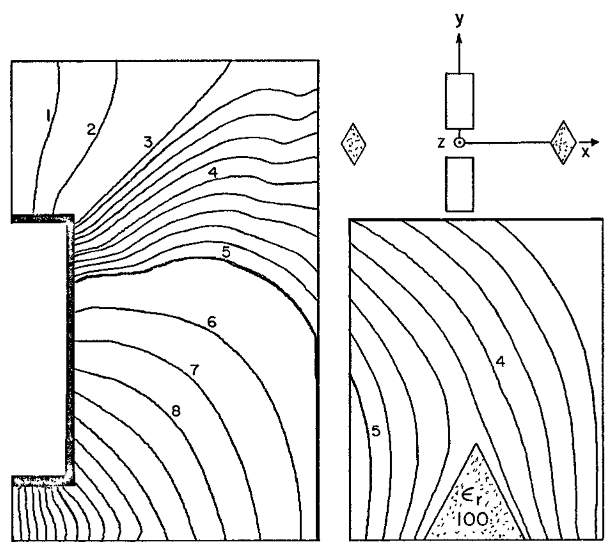

FIG. 4 AN ANTENNA PROBLEM WITH DIELECTRIC OBSTACLES AND SOLUTION USING TWO PICTURE-FRAMES